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Critical properties of the anisotropic Ising model on three-node hierarchical lattices

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Abstract. The criticality of the anisotropic Ising model on a three-node hierarchical lattice is investigated by an exact renormalisation group transformation. The phase diagram exhibits three physically different phases, namely a paramagnetic one and surface and bulk ferromagnetic ones. When $J_1 \neq J_2 \neq J_3$, the system orders in the direction with the largest J before it orders in the bulk. The bulk para-ferromagnetic transition is separated into three different universality classes, i.e. one isotropic and two anisotropic. This phenomenon, which is quite different from that on anisotropic Bravais lattices and seems analogous to that of a semi-infinite Ising model, is analysed.

1. Introduction

In recent years, much interest has been devoted to the study of phase transitions on hierarchical lattices (Berker and Ostlund 1979, Kaufman and Griffiths 1981, 1982, Hu 1985). Hierarchical lattices serve as a convenient theoretical laboratory where new ideas can be tested and developed, because of the performance of exact renormalisation group transformations, and provide insight into low-symmetry physical systems such as random magnets, interfaces, etc. Because of their highly inhomogeneous nature, it is very interesting to know the difference between critical phenomena on hierarchical and Bravais lattices. The work of Hu (1985) shows that critical properties on hierarchical lattices depend very much on the detailed structure of the lattice (an extended universality principle for these lattices may be refuted) which is quite different from the case of Bravais lattices where universality depends only on dimensionality and symmetry.

The anisotropic Ising model has been a subject of intense study in the past few years. Exact solutions of the model on the two-dimensional lattice (Onsager 1944; see also McCoy and Wu 1973) show that the critical temperatures decrease as the ratio $R = J_x/J_y$ decreases while the critical exponents remain identical for all finite non-zero values of R. Within a renormalisation group framework, it has been conjectured that spatial anisotropy is a marginal operator and that the critical behaviour should be described by a line of fixed points with critical exponents identical to those of the isotropic system (Bruce 1974, Aharony and Fisher 1980). What influence does spatial anisotropy have on critical phenomena in the low-symmetry lattices such as a hierarchical lattice and does it play the same role as on Bravais lattices? In the present paper, an anisotropic Ising model with nearest-neighbour interactions is investigated on a three-node hierarchical lattice which is introduced here. Our effort is dedicated

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to the analysis of the phase diagram, the various critical universality classes and the changes in critical phenomena as compared with anisotropic Bravais lattices, which might answer the above questions.

The outline of the paper is as follows. In § 2, the idea of multi-node hierarchical lattices is introduced and the construction of a three-node example is described. § 3 is devoted to the model and formalism and § 4 to the results. Finally, the conclusions are given in § 5.

2. Geometry

Hierarchical lattices are iteratively constructed by decorating an object with a basic cell, called a generator. Each generator of a bond studied so far has two nodes between which the decoration takes place. In these lattices, anisotropy of the lattice will vanish when the renormalisation group transformation takes place once. To study the anisotropic Ising model, we introduce the idea of multi-node hierarchical lattices. For instance, a three-node hierarchical lattice can be constructed by the generator of the two-dimensional Sierpinski gasket in a hierarchical way (see figure 1). When the length scale is changed by a factor 2, six new units are created. Therefore its fractal dimensionality (Mandelbrot 1977, 1982) is $D = \ln 6/\ln 2 = 2.5849\ldots$. The lattice is also a three-node hierarchical lattice (we call it L_2 for short) with the same fractal dimensionality as the orginal one (L_1) when the bonds in any one of the directions are absent and it is reduced to the collections of parallel two-node lattices of all sizes from a single site to a maximum when the bonds in any two directions are absent. Those two-node hierarchical lattices can be considered as the surfaces of the three-node one.

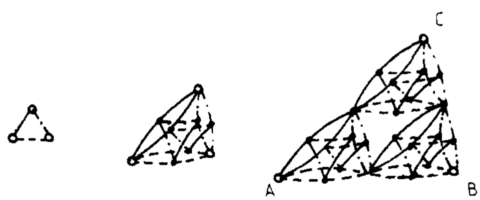

Figure 1. The initial triangle and the first two construction stages for the three-node hierarchical lattice. Only one half is shown in the second stage for simplicity and clearness; the whole should be made of two by connection at the points A, B and C (——, J_1; - - -, J_2; —·—, J_3).

3. Model and formalism

The anisotropic Ising model can be described by the Hamiltonian

$$H = -J_1 \sum_{ij} S_i S_j - J_2 \sum_{ik} S_i S_k - J_3 \sum_{il} S_i S_l \tag{1}$$

where J_1, J_2 and J_3 are respectively the nearest-neighbour exchange interactions along the three different directions on the three-node hierarchical lattice. The RG equations are obtained by summing over the internal spins of all the triangles of linear size 2.

The rescaling factor is $b = 2$. The resulting recursion relations for $K_i = J_i/kT$ ($i = 1, 2$ and 3) are

$$K_1 = \tfrac{1}{2}\ln\left(\frac{A}{B}\frac{\cosh(2K_1)}{\cosh(2K_2)\cosh(2K_3)}\right)$$
$$K_2 = \tfrac{1}{2}\ln\left(\frac{A}{B}\frac{\cosh(2K_2)}{\cosh(2K_1)\cosh(2K_3)}\right) \qquad (2)$$
$$K_3 = \tfrac{1}{2}\ln\left(\frac{A}{B}\frac{\cosh(2K_3)}{\cosh(2K_1)\cosh(2K_3)}\right)$$

with

$$\begin{aligned} A = {} & \exp(K_1+K_2+K_3)\cosh(2(K_1+K_2+K_3)) \\ & +\exp(K_1-K_2-K_3)\cosh(2(K_1-K_2-K_3)) \\ & +\exp(-K_1-K_2+K_3)\cosh(2(K_1+K_2-K_3)) \\ & +\exp(-K_1+K_2-K_3)\cosh(2(K_1-K_2+K_3)) \qquad (3) \\ B = {} & \exp(K_1+K_2+K_3)+\exp(K_1-K_2-K_3)+\exp(-K_1-K_2+K_3) \\ & +\exp(-K_1+K_2-K_3) \end{aligned}$$

which completely determine the RG recurrence in the parameter space.

4. Results

The RG flow diagram is shown in figure 2, only the part for $t_3 \geq t_2 \geq t_1$ ($t_i = \exp(-K_i)$, $i = 1, 2$ and 3) being indicated (the rest can be obtained easily in terms of the symmetry).

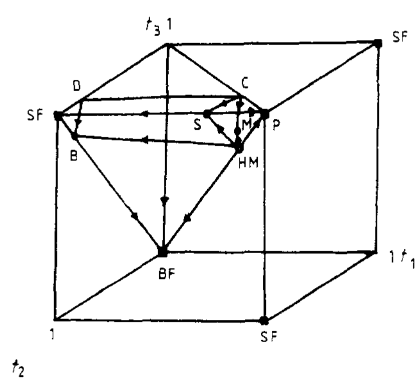

Figure 2. Flow diagram of the anisotropic Ising model on the three-node hierarchical lattice in t_1, t_2, t_3 space. Only the part satisfying $t_3 \geq t_2 \geq t_1$ is shown because of the symmetry. Trivial (stable), critical (semistable) and the multicritical (unstable) fixed points are denoted by ■, ● and ○, respectively. The paramagnetic and two ferromagnetic (surface and bulk) phases are separated by the C-S-HM and C-D-B-HM critical surfaces. Typical flowlines on these surfaces are shown.

It exhibits

(i) three different phases which are respectively characterised by three fully stable points at $(t_1, t_2, t_3) = (1, 1, 1)$ (paramagnetic phase, P), (0, 1, 1) (surface ferromagnetic phase, SF) and (0, 0, 0) (bulk ferromagnetic phase, BF);

(ii) a multicritical line in which parts which start at a high-order multicritical point (fully unstable fixed point) denoted HM and at point C (the intersection point of critical surface with the surface $t_3 = 1$) and end at the multicritical point (semistable fixed point) M;

(iii) three semistable (or critical) fixed points denoted, respectively, S corresponding to the infinite surface (two-node lattice), M for the bulk magnetic ordering and B for the bulk ordering after the surface has been ordered; and

(iv) four universality classes, the two-node hierarchical lattice (surface) one which occurs for $J_1 > J_2 \geq J_3$ with the critical temperature determined by the C-S-HM critical surface and three-node hierarchical lattice (bulk) ones which occur when $J_1 = J_2 = J_3$, $J_1 = J_2 > J_3$ and $J_1 > J_2 > J_3$ with the critical temperature determined by the fixed point HM, the C-M-HM critical line and the C-D-B-HM critical surface, respectively.

The locations of critical points and their exponents, which are accurate enough numerically to reveal the differences, can be found in table 1. The phase boundaries and a few representative flows for two subspaces which are closed under the flows are illustrated in figure 3. In the $(t_1, t(=t_2=t_3))$ space, the critical lines separating the P, SF and BF phases are indicated (figure 3(*a*)). When $t_1 > t$ $(J_1 < J)$, only an anisotropic bulk para-ferromagnetic transition, which does not belong to the isotropic universality class, is possible; when $t_1 < t$ $(J_1 > J)$, the surface orders before the bulk does and the critical exponents of the bulk transition are changed. The isotropic bulk para-ferromagnetic transition takes place when $t_1 = t$. The (t_2, t_3) space with $t_1 = 0$ $(J_1 = \infty)$ indicates

Table 1. Critical points and exponents for the 3NH model.

	(t_1, t_2, t_3)	(y_1, y_2, y_3)	(ϕ_1, ϕ_2)
Surface transition	(0.543 69, 1, 1)	(0.747 24, ,)	
Bulk transition	(0, 0.868 84, 0.868 84)	(0.839 25, ,)	
Multicritical point	(0.718 42, 0.718 42, 0.802 02)	(1.070 49, 0.212 45, −0.0803)	(0.198, −0.075)
High-order multicritical point	(0.746 12, 0.746 12, 0.746 12)	(1.077 30, 0.075 39, 0.075 39)	(0.070, 0.070)

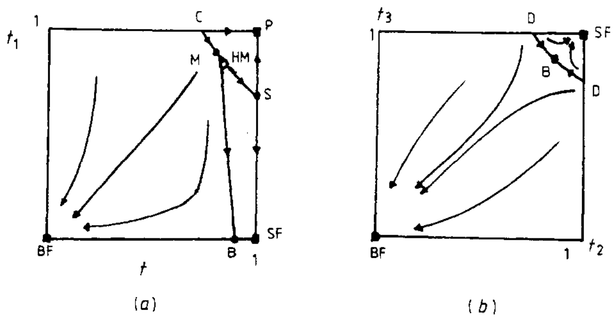

Figure 3. The phase boundaries and a few representative flows for two subspaces with (*a*) $t_2 = t_3$ $(J_2 = J_3)$ and (*b*) $t_1 = 0$ $(J_1 = \infty)$.

the critical line separating the P and BF phases, while the surface in the direction with the largest J (here J is J_1) has already ordered.

5. Conclusion

In the present paper, we have introduced and analysed an anisotropic Ising model on a novel three-node hierarchical lattice. The phase diagram is obtained via an exact decimation transformation method. The results of our study reveal some interesting features. The bulk para-ferromagnetic transition is separated into three different universality classes instead of one as in the case of Bravais lattices. As in the case of two-node hierarchical lattices (Hu 1985), Ising models with only one kind of exchange interaction on the L_1 and L_2 lattices do not belong to the same universality classes, although they have the same fractal and spectral dimensions (Rammal *et al* 1984), which may be due to the important difference that the L_2 lattices possess only reflection symmetry instead of the threefold rotation symmetry of the L_1 lattice. Another interesting feature that appears is that the infinite (maximum) two-node hierarchical lattice (surface) with the largest J orders before the three-node one (bulk) does even though the J in the other two directions is not zero. In view of this, the critical behaviour of the three-node hierarchical (3NH) model is rather more analogous to that of a semi-infinite Ising (SII) model than to that of an anisotropic Ising model on any standard lattice; this can be seen from the following correspondences between (multi) critical points of the 3NH phase diagram in the subspace $t_2 = t_3$ (figure 3(a)) and of the SII phase diagram (see e.g. Nakanishi and Fisher (1982) for a summary of results for the SII model):

3NH:	M	HM	S	B
SII:	Ordinary	Special	Surface	Extraordinary

The anisotropic crossover exponent at HM (see table 1) could be compared with the surface enhancement exponent at the special transition $\phi_1 = \frac{1}{2} + O(\varepsilon)$, $\varepsilon = 4 - d$. However, the largest eigenvalues at fixed points M, HM and B are different from each other and those at fixed points O, SP and E are the same. Namely the bulk para-ferromagnetic transition of the 3NH model is separated into three different universality classes instead of one as in the SII model. Despite some artificial and unphysical characters of the 3NH model, we feel that its rich phase structure may provide some insights into low-symmetry systems.

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